Part III PAPER-V NOTES-7

Algebra of limits

If f and g be two functions with a domain N, we define four functions $f \pm g$, fg, f/g on N by setting

(i.) $(f+g)(x,y) = f(x,y) + g(x,y)$

(ii.) $(f-g)(x,y) = f(x,y) - g(x,y)$

(iii.) $f \cdot g\,(x,y) = f(x,y) \cdot g(x,y)$

(iv.) $f/g\,(x,y) = f(x,y)/g(x,y)$ if

(v.) $g(x,y) \neq 0$ for $(x,y) \in N$

Th. 1 If f, g be two functions defined on some neighbourhood of a point (a,b) such that $\lim f(x,y) = l$, $\lim g(x,y) = m$ when $(x,y) \to (a,b)$ then

i) $\lim (f+g) = \lim f + \lim g = l+m$

ii) $\lim (f-g) = \lim f - \lim g = l-m$

iii) $\lim (f \cdot g) = \lim f \cdot \lim g = l \cdot m$

iv) $\lim \frac{f}{g} = \frac{\lim f}{\lim g} = \frac{l}{m}$

i) Proof Since $\lim\limits_{x \to c} f(x) = l$, $\lim\limits_{x \to c} g(x) = m$

therefore for any $\epsilon > 0$, $\exists$ positive numbers δ_1, δ_2 such that

$$|f(x) - l| < \frac{1}{2}\epsilon \text{ when } 0 < |x-c| < \delta_1$$
$$|g(x) - m| < \frac{1}{2}\epsilon \text{ when } 0 < |x-c| < \delta_2$$

If $\delta = \min(\delta_1, \delta_2)$, then for $0 < |x-c| < \delta$

$$|f(x) - l| < 1/2\,\epsilon, \; |g(x) - m| < 1/2\,\epsilon$$

and

$$|(f+g)(x) - (l+m)| = |f(x) - l + g(x) - m|$$
$$\leq |f(x) - l| + |g(x) - m|$$
$$< \epsilon$$

$$\Rightarrow |(f+g)x-(l+m)| < \epsilon \text{ when } 0<|x-c|<\delta.$$

$$\Rightarrow \lim_{x\to c} (f+g)x = l+m.$$

(ii) Proof is Similar to part(i)